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Harmonic Version of Jackson's Theorem in the Complex Plane

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The classical Jackson theorem concerning polynomial approximation of functions on $[-1, 1]$ is generalized to the case of approximation of functions given on a piecewise smooth arc in the complex plane by harmonic polynomials.

1. INTRODUCTION

Let L_0 denote the segment $[-1, 1]$ and let $f(x)$ be a function continuous on L_0. The famous theorem of Jackson states that for any integer $n \in \mathbb{N} := \{1, 2, ...\}$ there exists a polynomial $p_n(x)$ of degree at most n such that for any $x \in L_0$

$$|f(x) - p_n(x)| \leqslant c\omega_f(1/n), \tag{1.1}$$

where ω_f is the modulus of continuity of f and $c = \text{const} > 0$.

Many papers are devoted to generalizations of this statement by means of considering L_0 as a part of the complex plane $\mathbb{C}$ rather than merely part of the real line.

In particular, Newman [10] raised the question of whether (1.1) remains true when L_0 and $p_n(x)$ are replaced by some other Jordan arc $L \subset \mathbb{C}$ and algebraic polynomial $p_n(z)$ of the complex variable $z \in \mathbb{C}$. He proposed to say that a Jordan arc (briefly, arc) has the Jackson property (briefly, has (J)), if an analogy of (1.1) written for L remains true.

The problem of determining whether or not L has (J) turned out to be difficult. However, on the whole, thanks to the efforts of Newman [10],

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Andersson [1], Lesley [7], Mamedhanov [9], Maimeskul [8], and Anderson *et al.* [2], it seems to be solved.

In this work we suggest to study the same problem concerning a harmonic version of Jackson's theorem, i.e., approximation of a function f on an arc $L \subset \mathbb{C}$ by harmonic polynomials $t_n(z)$ of degree at most n.

We shall say that the arc L has the (JH)-property if a harmonic version of (1.1) remains true for L (for an exact definition see below).

Arcs possessing (J) and (JH) have a few similar properties. For example, repeating word for word a proof suggested by Newman [10, Theorem 1], one can show that if L has infinite length, then L does not have (JH).

Roughly speaking, the main purpose of this work is to show that the properties (J) and (JH) are essentially different. We reach this aim by proving that piecewise smooth arcs without cusps have (JH). At the same time, it is well known that such arcs need not have (J).

In reading the proofs in this paper the best example to have in mind is the arc consisting of two line segments meeting at a right angle. Note that in the paper [10] that started the discussion, Newman stated that he could not show that this arc does not have (J). This was only later shown to be the case by Andersson [1].

2. MAIN RESULT AND DEFINITIONS

Let $K \subset \mathbb{C}$ be a continuum and $\omega(\delta)$, $\delta > 0$, be a function of the type of modulus of continuity, i.e., a positive nondecreasing function (with $\omega(+0) = 0$), satisfying for some $c_1 = \text{const} \geqslant 1$ the condition

$$\omega(t\delta) \leqslant c_1 t\omega(\delta) \qquad (\delta > 0,\ t > 1). \tag{2.1}$$

We denote by $C^\omega(K)$ the class of all functions f continuous on K for which

$$|f(z_1) - f(z_2)| \leqslant \omega(|z_1 - z_2|) \qquad (z_1, z_2 \in K).$$

If $\omega(\delta) = c_2\delta$, $c_2 = \text{const} > 0$, then we shall use for $C^\omega(K)$ the notation $H^1(K, c_2)$.

The sets of analytic and, respectively, harmonic polynomials of degree at most $n \in \mathbb{N}$ are defined as follows:

$$\mathbb{P}_n := \left\{ p(z) = \sum_{j=0}^{n} a_j z^j : a_j \in \mathbb{C} \right\},$$

$$\mathbb{T}_n := \{ t(z) = \operatorname{Re} p(z) : p \in \mathbb{P}_n \}.$$

For $f \in C^{\omega}(K)$ set

$$E_n(f, K) := \inf_{p \in \mathbb{P}_n} \sup_{z \in K} |f(z) - p(z)|,$$

$$E_{n, \Delta}(f, K) := \inf_{t \in T_n} \sup_{z \in K} |f(z) - t(z)|.$$

We shall say that a Jordan arc $L \subset \mathbb{C}$ has (JH), if for any ω, any real-valued function $f \in C^{\omega}(L)$, and each $n \in \mathbb{N}$ the inequality

$$E_{n, \Delta}(f, L) \leqslant c_3 \omega(1/n) \tag{2.2}$$

holds with some constant $c_3 > 0$ depending only on L and c_1.

As usual, the Jordan arc L is of class C^{2+} (briefly, C^{2+}-smooth), if it has a parametrization $L = \{w(t) : 0 \leqslant t \leqslant 1\}$, where w is two times continuously differentiable and satisfies $w'(t) \neq 0$,

$$|w''(t_1) - w''(t_2)| \leqslant c_4 |t_1 - t_2|^{\alpha} \qquad (0 \leqslant t_1 < t_2 \leqslant 1)$$

with some constants $c_4 > 0$ and $0 < \alpha < 1$. The main result of this paper is the following statement.

THEOREM. *Any Jordan arc L consisting of a finite number of C^{2+}-smooth arcs without cusps has (JH).*

In what follows, we shall use the following notations:

$$D(z, \delta) := \{\zeta : |\zeta - z| < \delta\}, \qquad D := D(0, 1),$$

$$d(z, K) := \operatorname{dist}(z, K).$$

$f|_K$ denotes the restriction of f to K.

We denote by $A(K)$ the class of all functions continuous on K and analytic at its interior points. The symbol $\operatorname{Har}(K)$ will be used for the similar class of all real-valued functions continuous on K and harmonic at its interior points.

Furthermore, $c, c_1, \ldots$ denote positive constants, in general different at different occurrences and depending only on numbers that are not significant for the questions of interest. We shall also employ the symbols $a \preccurlyeq b$, denoting that $a \leqslant cb$, and $a \asymp b$, if simultaneously $a \preccurlyeq b$ and $b \preccurlyeq a$.

At last we need the notion of the second modulus of smoothness of a function $f \in A(K)$, which for our purposes is convenient to define as

$$\omega_2(f, \delta) := \sup_{z \in K} \inf_{p \in \mathbb{P}_1} \sup_{\zeta \in K \cap \overline{D(z, \delta)}} |f(\zeta) - p(\zeta)| \qquad (\delta > 0)$$

(see [4, 12 and 14]). Denote by $H_2^1(K, c)$ the class of all functions $f \in A(K)$ for which

$$\omega_2(f, \delta) \leqslant c\delta \qquad (\delta > 0).$$

3. SOME AUXILIARY RESULTS

For the proof of the theorem we need two special results concerning the properties of harmonic and analytic functions in domains with a piecewise smooth boundary.

Let the boundary $\Gamma = \partial G$ of a Jordan domain $G \subset \mathbb{C}$ consist of a finite number of C^{2+}-smooth arcs $\gamma_1, ..., \gamma_m$ which meet at the points $z_1, ..., z_m$ under the angles $\alpha_j \pi$ (with respect to G), $0 < \alpha_j < 1$.

Let g be a real continuous function on Γ. We use the same symbol for the harmonic extension of g to G. Further, we assume that $g|_\Gamma \in H^1(\Gamma, 1)$. Hence, comparing, for example, [6, Theorem 1] and [4, Theorem 3] we see that the analytic completion of g in G, i.e., the analytic function h satisfying $g = \operatorname{Re} h$ throughout G, can be extended continuously to Γ.

Even in the case of the unit disk the function h does not necessarily belong to $H^1(\bar{G}, c)$. However, we below establish a little weaker statement by using the notion of the second modulus of smoothness. This generalizes classical results of Hardy–Littlewood and Zygmund to piecewise C^{2+}-smooth curves, and is thus of independent interest.

LEMMA 1. *Let G, g, and h be as above. Then $h \in H_2^1(\bar{G}, c)$ with some $c = c(G)$.*

Proof. According to [4, Theorem 2] we only need to verify the inequality

$$|h''(\zeta)| \preccurlyeq 1/d(\zeta, \Gamma) \qquad (\zeta \in G). \tag{3.1}$$

In order to do this we introduce a conformal mapping φ of G onto D and set $w := \varphi(\zeta)$, $\psi := \varphi^{-1}$,

$$\tilde{g}(\tau) := (g \circ \psi)(\tau) \qquad (\tau \in \bar{D}),$$

$$\tilde{h}(\tau) := (h \circ \psi)(\tau) \qquad (\tau \in \bar{D}).$$

Without loss of generality we may assume that $\frac{1}{2} < |w| < 1$.

Next, we estimate from above the moduli of the first two derivatives of $\tilde{h}$ at w.

By Schwarz's formula,

$$\tilde{h}'(w)=\frac{1}{\pi}\int_{|\tau|=1}\frac{\tilde{g}(\tau)-\tilde{g}(w')}{(\tau-w)^2}\,d\tau, \tag{3.2}$$

$$\tilde{h}''(w)=\frac{2}{\pi}\int_{|\tau|=1}\frac{\tilde{g}(\tau)-\tilde{g}(w')}{(\tau-w)^3}\,d\tau, \tag{3.3}$$

where $w' := w/|w|$.

For definiteness, let z_1 denote the corner of Γ closest to ζ, i.e.,

$$|\zeta-z_1|=\min_{1\leqslant j\leqslant m}|\zeta-z_j|.$$

Setting $\alpha := \alpha_1$ (<1), we obtain by classical results due to Warschawski (cf. [11, Chap. 3])

$$|\varphi'(\zeta)|\asymp|\zeta-z_1|^{1/\alpha-1}, \tag{3.4}$$

$$|\varphi''(\zeta)|\preccurlyeq|\zeta-z_1|^{1/\alpha-2}, \tag{3.5}$$

$$1-|w|\asymp|\zeta-z_1|^{1/\alpha-1}\,d(\zeta,\Gamma), \tag{3.6}$$

$$|w-\tau_1|\asymp|\zeta-z_1|^{1/\alpha}, \tag{3.7}$$

where $\tau_1 := \varphi(z_1)$.

The next inequality is an immediate consequence of the distortion properties of the function φ: For $\tau\in\partial D$ and $z := \psi(\tau)$ we have

$$\frac{|\zeta-z|}{d(\zeta,\Gamma)}\preccurlyeq\left|\frac{w-\tau}{w-w'}\right|. \tag{3.8}$$

Indeed, first note that by (3.6) and (3.7)

$$d(\zeta,\Gamma)\asymp|w-w'|\,|w-\tau_1|^{\alpha-1}. \tag{3.9}$$

Now, we consider two cases.

(a) Let $|w-\tau|\leqslant\frac{1}{2}|w-\tau_1|$. Then

$$|\zeta-z|\asymp|w-\tau|\,|w-\tau_1|^{\alpha-1},$$

which taking into account (3.9) implies (3.8).

(b) Let $|w-\tau|>\frac{1}{2}|w-\tau_1|$. Then by (3.7)

$$\begin{aligned}|\zeta-z|&\leqslant|\zeta-z_1|+|z_1-z|\asymp|w-\tau_1|^{\alpha}+|\tau_1-\tau|^{\alpha}\\&\asymp|w-\tau|^{\alpha}.\end{aligned}$$

Therefore,

$$\frac{|\zeta - z|}{d(\zeta, \Gamma)} \preccurlyeq \frac{|w-\tau|^{\alpha}}{|w-w'|\,|w-\tau_1|^{\alpha-1}} \preccurlyeq \left|\frac{w-\tau}{w-w'}\right|,$$

and we obtain (3.8), too.

Set $\zeta' := \psi(w')$. For the function $\tilde{g}$ and $\tau \in \partial D$ we write according to (3.8)

$$\begin{aligned} |\tilde{g}(\tau) - \tilde{g}(w')| &= |g(z) - g(\zeta')| \leqslant |z - \zeta'| \\ &\preccurlyeq |\zeta - z| \preccurlyeq d(\zeta, \Gamma)\, \frac{|w-\tau|}{1-|w|}. \end{aligned}$$

Hence, by (3.3)

$$|\tilde{h}''(w)| \preccurlyeq \frac{d(\zeta, \Gamma)}{1-|w|} \int_{|\tau|=1} \frac{|d\tau|}{|\tau - w|^2} \preccurlyeq \frac{d(\zeta, \Gamma)}{(1-|w|)^2}. \tag{3.10}$$

To estimate $|\tilde{h}'(w)|$ from above, we divide the unit circle $J := \partial D$ into two parts, setting

$$J_1 := \{\tau \in J : |\tau - w| \leqslant 2\,|w - \tau_1|\}, \qquad J_2 := J \backslash J_1.$$

Further, by virtue of (3.6) and (3.7)

$$\begin{aligned} \left|\int_{J_1} \frac{\tilde{g}(t) - \tilde{g}(w')}{(\tau - w)^2}\, d\tau\right| &\preccurlyeq \frac{d(\zeta, \Gamma)}{1-|w|} \int_{J_1} \frac{|d\tau|}{|\tau - w|} \\ &\preccurlyeq \frac{d(\zeta, \Gamma)\,|w-\tau_1|}{(1-|w|)^2} \asymp \frac{|\zeta - z_1|}{1-|w|}. \end{aligned} \tag{3.11}$$

Since for $\tau \in J_2$ by (3.7)

$$\left|\frac{\zeta' - z}{\zeta - z_1}\right| \asymp \left|\frac{z - z_1}{\zeta - z_1}\right| \asymp \left|\frac{\tau - \tau_1}{w - \tau_1}\right|^{\alpha},$$

we have

$$\begin{aligned} \left|\int_{J_2} \frac{\tilde{g}(\tau) - \tilde{g}(w')}{(\tau - w)^2}\, d\tau\right| &\leqslant |\zeta - z_1| \int_{J_2} \left|\frac{z - \zeta'}{\zeta - z_1}\right| \frac{|d\tau|}{|\tau - w|^2} \\ &\asymp |\zeta - z_1|\, |w - \tau_1|^{-\alpha} \int_{J_2} \frac{|d\tau|}{|\tau - \tau_1|^{2-\alpha}} \preccurlyeq \left|\frac{\zeta - z_1}{w - \tau_1}\right|. \end{aligned} \tag{3.12}$$

Thus, combining (3.2), (3.11), and (3.12) we obtain

$$|\tilde{h}'(w)| \preccurlyeq \frac{|\zeta - z_1|}{1-|w|}. \tag{3.13}$$

Comparing (3.4)–(3.6), (3.10), (3.13), and the obvious relation

$$h''(\zeta) = \tilde{h}'(w)\,\varphi''(\zeta) + \tilde{h}''(w)(\varphi'(\zeta))^2,$$

we get (3.1). The proof is complete.

In the sequel, we shall also make use of the following elementary statement.

LEMMA 2. *Let G, g, and h be as above, and let $\gamma_1 \subset \gamma_2$ be subarcs of Γ whose end points do not coincide. If $g(z)=0$ for $z \in \gamma_2$, then for any ζ_1, $\zeta_2 \in \gamma_1$ the inequality*

$$|h(\zeta_1) - h(\zeta_2)| \leqslant c\,|\zeta_1 - \zeta_2|$$

holds with some constant $c = c(G, \gamma_1, \gamma_2)$.

Proof. As in the proof of Lemma 1 we consider the conformal mapping φ and the two auxiliary functions $\tilde{g}$ and $\tilde{h}$. Since by our assumption all angles of Γ are $<\pi$, we have for arbitrary points ζ_1 and $\zeta_2 \in \Gamma$

$$|\varphi(\zeta_1) - \varphi(\zeta_2)| \preccurlyeq |\zeta_1 - \zeta_2|.$$

Let now $\zeta_1, \zeta_2 \in \gamma_1$, $w_j := \varphi(\zeta_j)$, $j = 1, 2$. A simple computation involving Schwarz's formula shows that

$$\begin{aligned} |h(\zeta_1) - h(\zeta_2)| &= |\tilde{h}(w_1) - \tilde{h}(w_2)| \\ &\leqslant \frac{|w_1 - w_2|}{\pi} \int_{\partial D \setminus \varphi(\gamma_2)} \frac{|\tilde{g}(\tau)|\,|d\tau|}{|\tau - w_1|\,|\tau - w_2|} \preccurlyeq |w_1 - w_2| \preccurlyeq |\zeta_1 - \zeta_2|. \end{aligned}$$

4. PROOF OF THE THEOREM

It is sufficient to establish (2.2) only for $f \in H^1(L, 1)$. Indeed, if this particular case is proved, then for an arbitrary real-valued function $f \in C^{\omega}(L)$ we may reason as follows.

It is well known (cf. [13]) that the function f can be extended to $\mathbb{C}$ in such a way that the extended function, also denoted by f, belongs to the class $C^{c\omega}(\mathbb{C})$ with some constant $c \geqslant 1$ depending only on c_1 from (2.1).

Consider the function

$$f_n(z) := \frac{n^2}{\pi} \iint_{D(z,\, 1/n)} f(\zeta)\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta,\ z \in \mathbb{C}).$$

It is a known fact (see, for example, [5, Chap. IX]) that $f_n \in H^1(\mathbb{C}, c_2\omega(1/n)\, n)$ with c_2 depending on c from above and c_1 from (2.1).

Moreover,

$$|f_n(z) - f(z)| \leqslant c\omega(1/n) \qquad (z \in \mathbb{C}).$$

Therefore, the polynomial approximation of the function

$$\left.\frac{f_n}{c_2 n\omega(1/n)}\right|_L \in H^1(L, 1)$$

on L up to c_3/n, multiplied by $c_2 n\omega(1/n)$, will approximate the function f within $(c + c_2 c_3)\,\omega(1/n)$.

Thus, we henceforth assume that $f \in H^1(L, 1)$.

We recall that L consists of a finite number of C^{2+}-smooth arcs $l_1, ..., l_k$ with end points $z_0, ..., z_k$, respectively.

Fix some arcs $\gamma_j \subset l_j$ with end points ζ'_j and ζ''_j, not coinciding with z_{j-1} and z_j, and such that ζ''_j lies on l_j between ζ'_j and z_j.

Using intermediate polynomial interpolation of the function f at the points $\{\zeta'_j, \zeta''_j\}_{j=1}^k$ we reduce our problem to the approximation of a function f satisfying the following additional conditions:

$$f(\zeta'_j) = f(\zeta''_j) = 0 \qquad (j = 1, ..., k).$$

Below, we intend to use already known facts about polynomial approximation of functions on compact sets in $\mathbb{C}$ (presented, for example, in [5]). With respect to this theory, piecewise smooth arcs have in some sense an extremely bad structure, which is one of the main reasons why such arcs do not possess the (J)-property.

In the case of approximation by harmonic polynomials we are going to avoid this difficulty by presenting f as a sum of two functions. One of them will vanish identically in the neighborhood of $z_1, ..., z_{k-1}$, and the other will be extended harmonically to some auxiliary continuum $K \supset L$.

Namely, if the angle between l_j and l_{j+1} is not equal to π, we join the points ζ''_j and ζ'_{j+1} by a C^{2+}-smooth arc $s_j \subset \mathbb{C} \setminus L$ such that

(i) s_j and $l_j \cup l_{j+1}$ have the same tangents at points ζ''_j and ζ'_{j+1};

(ii) s_j together with the two subarcs of l_j and l_{j+1}, respectively, make up the boundary of some finite Jordan domain $e_j \subset \mathbb{C} \setminus L$ such that, with respect to this domain, ∂e_j has zero angles at ζ_j'' and ζ_{j+1}', and the angle $<\pi$ at z_j;

(iii) $\bar{e}_j \cap \bar{e}_m = \varnothing$ if $j \neq m$.

Consider the continuum

$$K := L \cup \bigcup_{j=1}^{k} \overline{e_j},$$

where we set $e_j := \varnothing$ in the case when the angle between l_j and l_{j+1} is equal to π.

Next, we cite, in a form convenient for us, two results concerning the polynomial approximation of harmonic and analytic functions on continua with a piecewise smooth boundary.

It is well known that the rate of polynomial approximation of functions from the classes $A(K)$ and $\mathrm{Har}(K)$ depends essentially on the metric properties of the conformal mapping Φ of the domain $\Omega := \hat{\mathbb{C}} \backslash K$ onto $\Delta := \hat{\mathbb{C}} \backslash \bar{D}$, where $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere, and Φ is normalized in such a way that $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$.

Metric properties of the function Φ for the case of continua with a piecewise smooth boundary have also been exhaustively investigated (see, for example, [11, Chap. 3]).

Hence, we see that the continuum K constructed above belongs, in the terminology of [5], for some fixed $m = m(K) \in \mathbb{N}$ to the class B_m^0 and, in the terminology of [3], to the class H^*. Moreover, for $z \in L$ we have

$$\rho_{1+1/n}(z) := \operatorname{dist}\left(z, \left\{\zeta : |\Phi(\zeta)| = 1 + \frac{1}{n}\right\}\right) \preccurlyeq \frac{1}{n}. \tag{4.1}$$

Taking into account results of [3] and (4.1) we get the following assertion.

Lemma 3. *For any $g \in H^1(K, 1) \cap \mathrm{Har}(K)$ and each $n \in \mathbb{N}$*

$$E_{n,\Delta}(g, K) \leqslant c/n, \qquad c = c(K).$$

Concerning approximation by analytic polynomials we need the result involving the notion of the second modulus of smoothness of a function $h \in A(K)$. Inequality (4.1) and the appropriate results of Dzyadyk [5, p. 443] and Shevchuk [12] allow us to formulate the following statement.

Lemma 4. *For any $h \in H_2^1(K, 1) \cap A(K)$ and each $n \in \mathbb{N}$*

$$E_n(h, K) \leqslant c/n, \qquad c = (K).$$

Next, we introduce the functions

$$f_1(z) := \begin{cases} f(z), & \text{if} \quad z \in \bigcup_{j=1}^{k} \gamma_j \\ 0, & \text{if} \quad z \in K \Big\backslash \bigcup_{j=1}^{k} \gamma_j. \end{cases}$$

$$f_2(z) := f(z) - f_1(z) \qquad (z \in L).$$

Since $f_1 \in H^1(K, c)$, by Lemma 3

$$E_{n, \Delta}(f_1, K) \preccurlyeq 1/n, \tag{4.2}$$

and we have to concentrate our efforts on the approximation of the function f_2.

Our next purpose is to construct a function h with the properties

$$h \in A(K) \cap H_2^1(K, c), \qquad \operatorname{Re} h|_L = f_2. \tag{4.3}$$

To do this, we join the points z_{j-1} and z_{j+1} (of course for j with $e_j \neq \varnothing$) by some C^{2+}-smooth arc $S_j \subset \Omega$ in such a way that a Jordan curve $R_j := S_j \cup l_j \cup l_{j+1}$ forms the boundary of a domain $E_j \supset e_j$ and R_j makes at the points z_{j-1}, z_j, and z_{j+1} inner angles $< \pi$ with respect to E_j.

Consider the function

$$f_{2,j}(z) := \begin{cases} f_2(z), & \text{if} \quad z \in \partial E_j \cap \partial e_j, \\ 0, & \text{if} \quad z \in \partial E_j \backslash \partial e_j. \end{cases}$$

We extend this function harmonically to E_j and denote by $h_j \in A(\bar{E}_j)$ the function satisfying $f_{2,j} = \operatorname{Re} h_j$ in E_j.

According to Lemma 1, $h_j \in H_2^1(\bar{E}_j, c)$. Moreover, if $\zeta_j''' \in \gamma_j$ is any fixed point, then, by virtue of Lemma 2, we can say even more about the smoothness of h_j in the neighborhood of ζ_j''',

$$h_j|_{l_j \cap \overline{D(\zeta_j''', \varepsilon)}} \in H^1(l_j \cap \overline{D(\zeta_j''', \varepsilon)}, c)$$

for sufficiently small $\varepsilon = \varepsilon(\zeta_j''') > 0$.

Since the functions h_j are defined up to an imaginary constant, we can adapt these constants so that the function

$$h(z) := \begin{cases} h_j(z), & \text{if} \quad z \in \bar{e}_j \cup L(\zeta_j''', \zeta_{j+1}'''),\ e_j \neq \varnothing \\ \text{piecewise imaginary constant elsewhere on } L, & \end{cases}$$

where $L(\zeta_j''', \zeta_{j+1}''')$ denotes the subarc of L between the respective points, belongs to the class $H_2^1(K, c)$.

Hence, we have constructed the desired function h satisfying (4.3), and using Lemma 4 we can write

$$E_{n,\Delta}(f_2, L) \leqslant E_n(h, K) \preccurlyeq 1/n. \tag{4.4}$$

Comparing (4.2) and (4.4), we get (2.2).

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